**Adjoint of Matrix**

**Definition.** If \( A \) is any \( n \times n \) matrix and \( C_{ij} \) is the cofactor of \( a_{ij} \), then the matrix

\[
\begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix}
\]

is called the **matrix of cofactors from** \( A \). The transpose of this matrix is called the **adjoint of** \( A \) and is denoted by \( \text{adj}(A) \).

Adjoint or Adjugate Matrix of a square matrix is the transpose of the matrix formed by the cofactors of elements of determinant \( A \).

To calculate adjoint of matrix we have to follow the procedure

a) Calculate Minor for each element of the matrix.

b) Form Cofactor matrix from the minors calculated.

c) Form Adjoint from cofactor matrix.

For an example we will use a matrix \( A \)

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

**Step 1:** Calculate Minor for each element.

To calculate the minor for an element we have to use the elements that do not fall in the same row and column of the minor element.

Minor of \( a_{11} = M_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = a_{22}a_{33} - a_{32}a_{23} \)
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Minor of \( a_{12} = M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23} \)

Minor of \( a_{13} = M_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22} \)

Minor of \( a_{21} = M_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13} \)

Similarly

\( M_{22} = a_{11}a_{33} - a_{31}a_{13} \) \quad \( M_{23} = a_{11}a_{32} - a_{31}a_{12} \)

\( M_{31} = a_{12}a_{23} - a_{22}a_{13} \) \quad \( M_{32} = a_{11}a_{23} - a_{21}a_{13} \) \quad \( M_{33} = a_{11}a_{22} - a_{21}a_{12} \)

**Step 2:** Form a matrix with the minors calculated.

\[
\text{Matrix of Minors} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}
\]

**Step 3:** Finding the cofactor from Minors:

**Cofactor:** A signed minor is called cofactor.

The cofactor of the element in the \( i^{th} \) row, \( j^{th} \) column is denoted by \( C_{ij} \)

\( C_{ij} = (-1)^{i+j} M_{ij} \)

Matrix of Cofactors = \begin{pmatrix} (-1)^{1+1} M_{11} & (-1)^{1+2} M_{12} & (-1)^{1+3} M_{13} \\ (-1)^{2+1} M_{21} & (-1)^{2+2} M_{22} & (-1)^{2+3} M_{23} \\ (-1)^{3+1} M_{31} & (-1)^{3+2} M_{32} & (-1)^{3+3} M_{33} \end{pmatrix}

Matrix of Cofactors = \begin{pmatrix} (1)M_{11} & (-1)M_{12} & (1)M_{13} \\ (-1)M_{21} & (1)M_{22} & (-1)M_{23} \\ (1)M_{31} & (-1)M_{32} & (1)M_{33} \end{pmatrix}
So we have

\[
\begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix} = \begin{pmatrix}
M_{11} & -M_{12} & M_{13} \\
-M_{21} & M_{22} & -M_{23} \\
M_{31} & -M_{32} & M_{33}
\end{pmatrix}
\]

**Step 4:** Calculate adjoint of matrix:

\[
\begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix}^T \Rightarrow \begin{pmatrix}
C_{11} & C_{21} & C_{31} \\
C_{12} & C_{22} & C_{32} \\
C_{13} & C_{23} & C_{33}
\end{pmatrix}
\]

To calculate adjoint of matrix, just put the elements in rows to columns in the cofactor Matrix. i.e convert the elements in first row to first column, second row to second column, third row to third column.

**EXERCISE 1**

1. let

\[
A = \begin{pmatrix}
3 & 2 & -1 \\
1 & 6 & 3 \\
2 & -4 & 0
\end{pmatrix}
\]

Find the Adjoint of A.

2. let

\[
B = \begin{pmatrix}
1 & 0 & -1 \\
3 & -1 & -2 \\
2 & 5 & 8
\end{pmatrix}
\]

Find the Adjoint of B

3. let

\[
C = \begin{pmatrix}
1 & 2 & 3 \\
-2 & 3 & -1 \\
-3 & 1 & 2
\end{pmatrix}
\]

Find the Adjoint of C
**Answers to Exercise 1**

1. \( \text{adj}(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix} \)

2. \( \text{adj}(B) = \begin{pmatrix} 2 & -5 & -1 \\ -28 & 10 & -1 \\ 17 & -5 & -1 \end{pmatrix} \)

3. \( \text{adj}(C) = \begin{pmatrix} 7 & -1 & -11 \\ 7 & 11 & -5 \\ 7 & -7 & 7 \end{pmatrix} \)